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Generalized Darboux transformations: classification of inverse scattering methods for the radial Schrödinger equation*

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Abstract. A wide class of Darboux transformations of a Sturm–Liouville equation providing a unified treatment of exactly solvable models in quantum scattering theory is considered systematically. A classification of Darboux transformations is given and the relations between Darboux transformations and standard inverse scattering procedures for the radial Schrödinger equation including the matrix method of Newton–Sabatier are exhibited. In particular, the definitions, properties and matrix generalizations of Darboux transformations associated with Marchenko-type integral equations are studied in detail.

1. Introduction

Darboux transformations, originally introduced in a theorem on second-order differential equations [1], represent a powerful tool in generating families of exactly solvable Hamiltonians. They allow controlled manipulations of the spectrum and are therefore closely related with supersymmetric quantum mechanics and inverse problems in quantum scattering theory. In particular, Darboux transformations yield exactly solvable models which are used to generate a local, spherically symmetric potential from a given S-matrix [2–5]. Such exactly solvable models have played an important role in clarifying the restricted uniqueness of the solutions of inverse scattering problems with spherically symmetric potentials [6, 7]. Today they are at least of equal importance because applications of inverse scattering techniques to experimental data are almost exclusively based on them [8–12]. Darboux transformations, which are directly related to Bäcklund transformations [13], have also become an essential ingredient in the study of nonlinear partial differential equations [14, 15].

Darboux transformations provide closed mathematical relations between the transformed Hamiltonian and a background Hamiltonian, where all features of the latter are completely known but not necessarily in closed mathematical form. The S-matrices associated with these transformations are of simple rational structure and are flexible enough to represent approximately a large class of experimentally given S-matrices. The exactly solvable models generated by Darboux transformations are special solutions of the integral equations of

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inverse scattering theory [16, 17]. The general solutions of the inverse scattering schemes are not restricted to special rational forms of the S-matrix. However, they have never been applied to data in their full generality because the determination of the required spectral kernel from the S-matrix is very tedious. In contrast to this the approximation of experimental data by S-matrices of exactly solvable models is rather simple and has been used in almost all applications of inverse scattering techniques. Furthermore, in exactly solvable models Coulombic terms in the interaction can easily be included via an appropriate choice of the background potential.

Following Rudyak and Zakhariev [18] we have recently [5] studied a specific class of Darboux transformations of the Sturm–Liouville problem

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \frac{\lambda_0^2 - \frac{1}{4}}{r^2} + k_0^2 - V(r)\right)\psi(\theta, r) = \theta^2 h(r)\psi(\theta, r) \tag{1}$$

where the potential $V(r)$ and the constants θ, k_0 and λ_0 are, in general, complex. This is a slightly more general problem than the radial Schrödinger equation, allowing a unified treatment of the standard inverse scattering problem at fixed energy and at fixed angular momentum. The physical meaning of the continuous variable θ depends on the specific choice of the function $h(r)$ which is assumed to be not equal zero in the domain considered. An inverse scattering problem of the radial Schrödinger equation at fixed angular momentum is obtained by setting $h(r) = -1 + h_V(r)$, where $h_V(r)$ fulfils the conditions of a potential in quantum scattering theory (see e.g. [16, chapter I]). In this case θ is associated with the corresponding wavenumber k,

$$k^2(\theta) = k_0^2 + \theta^2. \tag{2}$$

Alternatively, one obtains the inverse scattering problem of the radial Schrödinger equation at fixed energy by the choice $h(r) = 1/r^2 + h_V(r)$ with the angular momentum $\lambda(\theta)$ given by

$$\lambda^2(\theta) = \lambda_0^2 + \theta^2\,. \tag{3}$$

The Sturm–Liouville ansatz also includes new inverse scattering problems where λ^2 and k^2 fulfil a linear relationship [5, 18].

In the previous paper [5] an important symmetry property of Darboux transformations has been derived and exploited to demonstrate how general integral equations can be obtained from Darboux transformations. These integral equations yield the Darboux transformations as solutions for degenerate kernels. It turns out that the integral equation of Gel'fand–Levitan [16, 17] represents only one possibility for formulating the inverse scattering theory. It has, however, the advantage that in this specific formulation the available spectral information can be implemented.

The type of Darboux transformations discussed in [5] leads to a wide class of exactly solvable models comprising those of Theis [19] and Lipperheide and Fiedeldey [2, 3]. These transformations do not cover all inverse scattering schemes, and the associated integral equations are limited to those of Gel'fand–Levitan type. There also exist other inverse scattering techniques, e.g. the matrix method of Newton and Sabatier [16, 20], which are similar in structure but could not be derived in our previous study [5]. Thus the question arises of whether one can establish further Darboux transformations of (1) which are not in the class discussed in [5], and which are related to other such inverse scattering schemes.

In the present paper we systematically extend the transformations of a Sturm–Liouville equation, originally discussed by Rudyak and Zakhariev [18] from the inverse scattering point of view. In addition to the type of Darboux transformation studied in [5] we consider a second type of Darboux transformation of (1) and formulate its matrix generalization. A careful investigation reveals that both types of Darboux transformations together yield a complete description of generalized integral equations with degenerate kernels. Thus we are able to give a classification of Darboux transformations as well as a demonstration of the inter-relations between various inverse scattering procedures.

In section 2 we give a brief survey of the Darboux transformations of (1) and their matrix generalizations. In particular, we define a second type of Darboux transformation and transformations related to the former by the symmetry property established in [5]. Their associated integral equations as well as their inter-relations are discussed in section 3. Combining the present results and those of [5] we can demonstrate the relation between inverse scattering schemes based on Darboux transformations and those using standard integral equations. This is discussed in section 4, where it is shown, in particular, that the matrix method of Newton and Sabatier [16, 20] can be regarded as a special set of Darboux transformations. In addition, as a by-product of our study, we obtain new exactly solvable models. Finally, in section 5 we give a brief summary.

2. Darboux transformations

In this paper we focus our attention on Darboux transformations of the Sturm–Liouville equation (1) and their application to the construction of exactly solvable models. In particular we aim at presenting a complete class of transformations in a rather compact way, in order to emphasize their common properties.

In the following we assume that we know the whole set of solutions $\psi_0(\theta, r)$ of the differential equation (1) with the potential $V_0(r)$. A simple Darboux transformation associated with a specific solution $\zeta_0(\alpha, r)$ of (1) with fixed $\theta = \alpha$ can be written in the form (cf [5, equation (2)])

$$\psi_1(r) = h^{-1/4}(r)\frac{A(r)}{\zeta_0(\alpha, r)} \tag{4}$$

with $A(r)$ characterizing the transformation. We distinguish two types

$$\text{(I)} \quad A^{\mathrm{I}}(r) = Bh^{-1/4}(r)W[\eta_0(\gamma, r); \zeta_0(\alpha, r)] \qquad \psi_1^{\mathrm{I}}(r) \to \eta_1(\gamma, r) \tag{5}$$

$$\text{(II)} \quad A^{\mathrm{II}}(r) = Bh^{-1/4}\left(1 + C\int_c^r \mathrm{d}s\, h(s)\zeta_0^2(\alpha, s)\right) \qquad \psi_1^{\mathrm{II}}(r) \to \zeta_1(\alpha, r) \tag{6}$$

where $\eta_0(\gamma, r)$ is a solution of (1) with the potential $V_0(r)$ and $\theta = \gamma$. The quantities B and C are arbitrary constants ($B \neq 0$) independent of r but may depend on the parameters of the transformations (e.g. α, γ). The first type, $\psi_1^{\mathrm{I}}(r)$, can be considered as the transformation of $\eta_0(\gamma, r)$ via the Wronskian

$$W[\eta_0(\gamma, r); \zeta_0(\alpha, r)] = \eta_0(\gamma, r)\left(\frac{\mathrm{d}}{\mathrm{d}r}\zeta_0(\alpha, r)\right) - \left(\frac{\mathrm{d}}{\mathrm{d}r}\eta_0(\gamma, r)\right)\zeta_0(\alpha, r) \tag{7}$$

and we therefore denote it by $\eta_1(\gamma, r)$. The second type involves only $\zeta_0(\alpha, r)$ and the integration limit c. In this case the function $\psi_1^{\mathrm{II}}(r)$ represents a transformation of $\zeta_0(\alpha, r)$ and we therefore use the notation $\zeta_1(\alpha, r)$ for it. For a specific choice of the constants B and C the transformation (6) can be considered as a degenerate limit of (5).

Both types of transformations yield solutions $\psi_1(r)$ of the Sturm–Liouville equation (1) with the same potential $V_1(r)$,

$$V_1(r) = V_0(r) - 2h^{1/2}(r)\frac{\mathrm{d}}{\mathrm{d}r}\left(h^{-1/2}(r)\frac{\mathrm{d}}{\mathrm{d}r}\ln(h^{1/4}(r)\zeta_0(\alpha, r))\right). \tag{8}$$

In principle the transformation (4) and the potential (8) are only well defined for r-values with non-vanishing $\zeta_0(\alpha, r)$. Hence, reasonable transformations of the form (4), (8) can only be derived from solutions $\zeta_0(\alpha, r)$ which do not vanish on the whole domain. Whether $\zeta_0(\alpha, r)$ fulfils this requirement depends mainly on the choice of the solution and the value of α, while the potential plays a minor role. In general, for physically reliable potentials there exist such solutions at proper α values. However, this restriction on $\zeta_0(\alpha, r)$ is not a severe limitation for applications because in the iterated forms, discussed subsequently, cancellations of the singularities might occur [21] and there exist many examples where the total transformation can be defined on the half or even on the full line [2–4,16,19]. At this point it is opportune to remark that transformations of type I associated with $\zeta_0(\alpha, r)$ yield solutions $\eta_1(\gamma, r)$ for each value of the continuous variable $\theta = \gamma$, whereas in the transformations of type II there is only one solution $\zeta_1(\alpha, r)$ for the specific value $\theta = \alpha$.

The Darboux transformations (4)–(6) provide us with the fundamental tool for constructing exactly solvable models of the radial Schrödinger equation. In [5] we have extensively studied Darboux transformations of type I showing their properties and their relations to several well known inverse scattering methods like the Gel'fand–Levitan equation [16], the inversion procedure of Theis [19], and the schemes developed by Lipperheide and Fiedeldey [2, 3]. In the following we deal with transformations of type II in more detail.

The definition (4)–(6) gives only the simplest Darboux transformations which are used as basic building blocks. Further transformations can be derived easily using the following general property of Darboux transformations. Let us take any solution $\xi_1(\gamma, r)$ of the Sturm–Liouville problem (1) at $\theta = \gamma$ with the potential

$$V_1(r) = V_0(r) - 2h(r)^{1/2}\frac{\mathrm{d}}{\mathrm{d}r}\left(h(r)^{-1/2}\frac{\mathrm{d}}{\mathrm{d}r}\ln \tilde{B}(r)\right) \tag{9}$$

which can be written depending on two functions $\tilde{A}(r)$ and $\tilde{B}(r)$ as

$$\xi_1(\gamma, r) = h(r)^{-1/4}\frac{\tilde{A}(r)}{\tilde{B}(r)}. \tag{10}$$

It can be shown [5] that there exists a further transformation

$$\tilde{\xi}_1(\gamma, r) = h(r)^{-1/4}\frac{\tilde{B}(r)}{\tilde{A}(r)} \tag{11}$$

which gives a solution of the Sturm–Liouville problem (1) with the potential

$$V_1(r) = V_0(r) - 2h(r)^{1/2}\frac{\mathrm{d}}{\mathrm{d}r}\left(h(r)^{-1/2}\frac{\mathrm{d}}{\mathrm{d}r}\ln \tilde{A}(r)\right). \tag{12}$$

Analogous to the simple transformations (4) and (8) the associated transformations (11) and (12) are only well defined for r values with non-vanishing $\tilde{A}(r)$.

Applying this symmetry property of Darboux transformations to the transformations (6) we immediately obtain the function

$$\zeta_2(\alpha, r) = \frac{1}{B}\,\frac{\zeta_0(\alpha, r)}{1 + C\int_c^r \mathrm{d}s\, h(s)\zeta_0^2(\alpha, s)} \tag{13}$$

which is a solution of (1) with the potential $V_2(r)$,

$$V_2(r) = V_0(r) - 2h^{1/2}(r)\frac{\mathrm{d}}{\mathrm{d}r}\left(h^{-1/2}\frac{\mathrm{d}}{\mathrm{d}r}\ln\left(1 + C\int_c^r \mathrm{d}s\, h(s)\zeta_0^2(\alpha, s)\right)\right). \tag{14}$$

Further Darboux transformations can be constructed by iteration, e.g. another solution of (1) with the same potential $V_2(r)$ is given by

$$\begin{aligned}\eta_2(\gamma, r) &= h^{-1/4}(r)\frac{h^{-1/4}(r)W[\eta_1(\gamma, r); \zeta_1(\alpha, r)]}{\zeta_1(\alpha, r)} \\ &= \eta_0(\gamma, r) + \frac{C}{\gamma^2-\alpha^2}\frac{W[\eta_0(\gamma, r); \zeta_0(\alpha, r)]}{1 + C\int_c^r \mathrm{d}s\, h(s)\zeta_0^2(\alpha, s)}\zeta_0(\alpha, r)\,.\end{aligned} \tag{15}$$

This transformation (15) is an iteration of the Darboux transformations (4)–(6) where the transformation of type II associated with $\zeta_0(\alpha, r)$ is followed by one of the first type associated with the transformed function $\zeta_1(\alpha, r)$. Both transformations $\zeta_2(\alpha, r)$ and $\eta_2(\gamma, r)$ play an essential role with respect to applications in inverse scattering problems.

In order to obtain Darboux transformations of sufficient complexity to be suitable for fitting experimental S-matrices we must employ matrix generalizations in analogy to [5]. A straightforward extension of (15) leading to more general Darboux transformations which depend on N solutions $\zeta_0(\alpha_1, r), \ldots, \zeta_0(\alpha_N, r)$ is obtained by defining

$$\eta_2(\gamma, r) = \eta_0(\gamma, r) - X^T(r)Y^{-1}(r)\zeta_0(r) \tag{16}$$

with vectors

$$X(r) = \begin{pmatrix} \dfrac{W[\eta_0(\gamma, r); \zeta_0(\alpha_1, r)]}{\gamma^2-\alpha_1^2} \\ \vdots \\ \dfrac{W[\eta_0(\gamma, r); \zeta_0(\alpha_N, r)]}{\gamma^2-\alpha_N^2} \end{pmatrix} \qquad \zeta_0(r) = \begin{pmatrix} \zeta_0(\alpha_1, r) \\ \vdots \\ \zeta_0(\alpha_N, r) \end{pmatrix} \tag{17}$$

and the N-dimensional matrix

$$Y(r) = \begin{pmatrix} C_1 - \displaystyle\int_c^r h(s)\zeta_0^2(\alpha_1, s)\mathrm{d}s & \cdots & \dfrac{W[\zeta_0(\alpha_1, r); \zeta_0(\alpha_N, r)]}{\alpha_1^2-\alpha_N^2} \\ \vdots & \ddots & \\ \dfrac{W[\zeta_0(\alpha_N, r); \zeta_0(\alpha_1, r)]}{\alpha_N^2-\alpha_1^2} & \cdots & C_N - \displaystyle\int_c^r h(s)\zeta_0^2(\alpha_N, s)\mathrm{d}s \end{pmatrix}. \tag{18}$$

Here, T denotes the transpose of a vector or matrix. The function $\eta_2(\gamma, r)$ of (16) is a solution of the Sturm–Liouville equation (1) with the potential $V_2(r)$,

$$V_2(r) = V_0(r) - 2h^{1/2}(r)\frac{\mathrm{d}}{\mathrm{d}r}h^{-1/2}(r)\frac{\mathrm{d}}{\mathrm{d}r}\ln\det Y\,. \tag{19}$$

Similarly, a matrix generalization of (13) leading to a solution associated with the potential (19) can be obtained,

$$\zeta_2(r) = Y^{-1}\zeta_0(r) \tag{20}$$

where the vector $\zeta_2(r)$ is defined in analogy to $\zeta_0(r)$ in (17).

From the formal point of view the matrix generalizations (16)–(20) look quite similar to those obtained for the type I transformations in [5]. However, it should be emphasized that (16)–(20) depend only on the N functions $\zeta_0(\alpha_1, r), \ldots, \zeta_0, (\alpha_N, r)$, whereas the corresponding generalization in [5] depends on $2N$ solutions of (1) of the potential $V_0(r)$.

3. Integral representations

We have shown in [5] that starting from Darboux transformations of type I we can construct general integral equations which are of the same form as those obtained in certain standard inverse scattering theories. We were able to identify specific Darboux transformations with solutions of the Gel'fand–Levitan procedure [16] and the integral equation of Burdet *et al* [23]. Other formulations of inverse scattering theories [16], e.g. the Marchenko method or the matrix method of Newton and Sabatier, could not be associated with Darboux transformations of type I.

We now have the second type of transformations at our disposal, and we consider in the following the transition from Darboux transformations (16)–(18) to integral equations. As in [5] we introduce the condition

$$W[\zeta_0(\alpha_i, r); \zeta_0(\alpha_j, r)]_{r=c} = 0 \qquad \forall i, j\,. \tag{21}$$

We can then cast (16)–(18) into the form

$$\eta_2(\gamma, r) = \eta_0(\gamma, r) + \int_c^r \mathrm{d}s\, h(s) K(r, s) \eta_0(\gamma, s)\,. \tag{22}$$

The kernel $K(r, s)$ is degenerate

$$K(r, s) = \zeta_2^T(r)\zeta_0(s) \tag{23}$$

and satisfies the intertwining relation

$$D_2(r)K(r, s) = D_0(s)K(r, s) \tag{24}$$

where D_0 and D_2 denote the differential operators

$$D_i(r) = \frac{1}{h(r)}\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \frac{\lambda_0^2 - \frac{1}{4}}{r^2} + k_0^2 - V_i(r)\right) \qquad i = 0, 2\,. \tag{25}$$

The function $K(r, s)$ exhibits all the features of a generalized translation kernel [16, 22] and can be obtained from the integral equation

$$K(r, t) = Q(r, t) - \int_c^r \mathrm{d}s\, h(s) K(r, s) Q(s, t) \qquad c \leqslant t \leqslant r \quad \text{or} \quad r \leqslant t \leqslant c \tag{26}$$

where $Q(r, t)$ is given as

$$Q(r, t) = \sum_{i=1}^{N} \zeta_0(\alpha_i, r)\, \frac{1}{C_i}\, \zeta_0(\alpha_i, t) \tag{27}$$

and satisfies the relation of a symmetric kernel

$$D_0(r)Q(r, t) = D_0(t)Q(r, t)\,. \tag{28}$$

Thus Darboux transformations of both types are associated with different degenerate kernels of the same integral equation.

In order to understand the inter-relations and the degree of completeness of the Darboux transformations (4)–(6) we must base our considerations on the common integral equation (26). Both transformations correspond to degenerate kernels $Q(r, t)$ and $K(r, t)$. In principle there are only two possibilities to construct a degenerate symmetric kernel $Q(r, t)$

using solutions of (1) with the potential $V_0(r)$. The trivial one uses only N solutions $\zeta_0(\alpha_i, r), i = 1, 2, \ldots, N$,

$$Q(r,t) = \sum_{i=1}^{N} \zeta_0(\alpha_i, r)\, d_i\, \zeta_0(\alpha_i, t)\,. \tag{29}$$

It is straightforward to show that this choice of $Q(r,t)$ leads directly to Darboux transformations of type II.

The second possibility to form a degenerate kernel $Q(r,t)$ requires $2N$ solutions $\eta_0(\gamma_i, r), \zeta_0(\alpha_i, r), i = 1, 2, \ldots, N$. Because of the intertwining relation (24) for the transformation kernel $K(r,t)$ the form of $Q(r,t)$ is, in this case,

$$Q(r,t) = \begin{cases} \displaystyle\sum_{i=1}^{N} \eta_0(\alpha_i, r)\, d_i\, \zeta_0(\alpha_i, t) & c \leqslant r \leqslant t \text{ or } c \geqslant r \geqslant t \\ \displaystyle\sum_{i=1}^{N} \eta_0(\alpha_i, t)\, d_i\, \zeta_0(\alpha_i, r) & c \leqslant t \leqslant r \text{ or } c \geqslant t \geqslant r\,. \end{cases} \tag{30}$$

Evaluation of the integral equation with the symmetric kernel (30) yields Darboux transformations of type I outlined in [5].

Apart from a combination of the expressions (29) and (30) there exists no further possibility to construct degenerate kernels $Q(r,t)$ with the solutions of the background problem. Hence the Darboux transformations (4)–(6) form, in this sense, a complete basis set associated with degenerate symmetric kernels $Q(r,t)$.

4. Relationships between inverse scattering methods

The identification of Darboux transformations with specific functional forms of the symmetric kernel of the common basic integral equation offers the possibility of revealing the relationships between different inverse scattering theories and exactly solvable models. The relations are not transparent in as much as for a given inverse scattering problem several formulations of inverse scattering theories are possible, such as, for example, the schemes for the inverse scattering problem at fixed angular momentum by Gel'fand–Levitan and Marchenko. In the following we concentrate on the counterparts of Darboux transformations of type II.

First, let us consider the inverse scattering problem at fixed angular momentum, $h(r) = -1$. Setting $c = \infty$ and choosing Jost functions $\zeta_0(\alpha_i, r) = f_0(\alpha_i, r), i = 1, 2, \ldots, N$ in the symmetric kernel (29), the integral equation (26) reduces to that of the Marchenko formalism. It is obvious that the solutions of the Marchenko equation are given by the Darboux transformations (16) associated with the Jost solutions. The corresponding potential is given by (19).

A remarkable feature of Darboux transformations of type II is the possibility to formulate matrix methods. An equation appropriate for a matrix method associated with the set of solutions $\{\zeta_0(\alpha_1, r), \ldots, \zeta_0(\alpha_N, r)\}$ can easily be obtained from the transformation (16). Applying the transformation only for the specific set of γ-values, $\gamma \in \{\alpha_1, \alpha_2, \ldots, \alpha_N\}$, and with the choice $\eta_0(\alpha_i, r) = \zeta_0(\alpha_i, r)$ we obtain the system of equations

$$\bar{\zeta}_2(\alpha_i, r) = \zeta_0(\alpha_i, r) + \int_c^r \mathrm{d}s\, h(s)\zeta_0(\alpha_i, s)\zeta_0^T(s)Y^{-1}(r)\zeta_0(r) \qquad i = 1, 2, \ldots, N\,. \tag{31}$$

For further manipulations it is convenient to rewrite (31) as a matrix equation

$$\bar{\zeta}_2(r) = \zeta_0(r) + L(r)Y^{-1}(r)\zeta_0(r) \tag{32}$$

defining the N-dimensional matrix

$$L(r) = (L_{ij}(r)) = \left(\int_c^r \mathrm{d}s\, h(s)\zeta_0(\alpha_i, s)\zeta_0(\alpha_j, s) \right). \tag{33}$$

The vector of the transformed functions is defined analogously to $\zeta_0(r)$ in (16) and denoted by $\bar{\zeta}_2(r)$ in order to distinguish it from the transformation (20). Within our restricted assumptions the matrix Y can be expressed through L via

$$Y = C \dot{-} L \tag{34}$$

where we have introduced the diagonal $N \times N$ matrix C with the non-vanishing matrix elements C_i, $i = 1, 2, \ldots, N$, given in (18). Using (34) together with the transformation (20) we can establish the relationship

$$\bar{\zeta}_2(r) = C\zeta_2(r)\,. \tag{35}$$

Thus we are able to cast (32) into the form

$$\zeta_2(r) = C^{-1}\zeta_0(r) + C^{-1}L(r)\zeta_2(r) \tag{36}$$

which represents the basic equation of a generalized matrix method [16]. In (36) the spectral information enters via the solutions $\zeta_2(r)$. Because it is directly derived from (16)–(18) it is obvious that the resulting potential belongs to the class of potentials associated with Darboux transformations of type II.

Equation (36) has been obtained for a general Sturm–Liouville equation and an arbitrary set of solutions $\zeta_0(\alpha_1, r), \ldots, \zeta_0(\alpha_N, r)$. Of particular interest is the application of (36) for the inverse scattering problem at fixed energy, $h(r) = 1/r^2$. Choosing $c = 0$ and using regular solutions $\phi_i(\alpha_i, r)$ for the function $\zeta_0(\alpha_i, r)$, equation (36) reduces to the basic equation of the Newton–Sabatier method [16]. However, it should be remarked that in this derivation only finite N -values have been considered, whereas in the matrix method of Newton–Sabatier the existence of the solution of (36) has been proved for an infinite sum.

Combining the results of the present work with those of [5] we can draw up a unified scheme indicating the relation between standard inverse scattering theories on the half line and exactly solvable models based on Darboux transformations (table 1). In the centre column of table 1 we show the two types of Darboux transformations of (1) and indicate the form of the associated symmetric kernel and the integration range in a schematic way. Below, the corresponding exactly solvable models are indicated with the names of their authors. In the left and right columns of table 1 we have listed the well established inverse scattering theories at fixed energy and fixed angular momentum, respectively. It is interesting to note that all inverse scattering theories for the radial Schrödinger equation reduce for degenerate symmetric kernel Q to exactly solvable models based on Darboux transformations.

Our scheme (table 1) indicates a lack of exactly solvable models for Darboux transformations of type II. Such new models can immediately be constructed from the transformation (16) given in section 2. Let us consider a general problem with $h(r) = a/r^2 + b + h_V(r)$, where a and b are non-vanishing complex numbers and $h_V(r)$ fulfils the criteria of a potential in scattering theory. Furthermore, we use Jost solutions for $\zeta_0(r)$.

Table 1. Scheme indicating the relations between generalized Darboux transformations and standard inverse scattering theories. The form of the symmetric kernel Q, as well as the range of integration of the integral equations, are given schematically. Well established exactly solvable models are also cited. For further discussion see text.

Fixed energy $h(r) = 1/r^2$	General inverse scattering problem $h(r) = a + b/r^2 + h_V(r)$	Fixed angular momentum $h(r) = -1$
Burdet it et al [23]	type I	Gel'fand–Levitan [16, 17]
$f_0^T \phi_0$, $\int_r^\infty$	$Q = \eta_0^T \zeta_0$, $\int_c^r$	$f_0^T \phi_0$, $\int_0^r$
Lipperheide–Fiedeldey [2, 3]	Schnizer–Leeb [5]	Theis–Bargmann [19, 6, 7]
	type II	Marchenko [16]
$\phi_0^T \phi_0$, $\int_0^r$	$Q = \zeta_0^T \zeta_0$, $\int_c^r$	$f_0^T f_0$, $\int_r^\infty$
Newton–Sabatier [16]	Rudyak–Zakhariev [18]	

From the asymptotic behaviour of the transformed regular function (16) one can determine the corresponding S-matrix

$$S(\gamma) = S_0(\gamma) \prod_{i=1}^{N} \frac{(k(\gamma) - k(\alpha_i))^2}{(k(\gamma) + k(\alpha_i))^2} \tag{37}$$

where $k(\gamma)$ is given by

$$k^2(\gamma) = k_0^2 - b\gamma^2 \tag{38}$$

and $S_0(\gamma)$ is the S-matrix associated with the background potential $V_0(r)$.

5. Summary

We have considered a set of Darboux transformations of a Sturm–Liouville equation which allows a treatment of a general inverse scattering problem with a linear dependence of λ^2 and k^2 (2), (3). Extending our previous work [5] we have studied a second type of Darboux transformation associated with only one solution $\zeta_0(\alpha, r)$. Twice iterated Darboux transformations as well as matrix generalizations have been obtained which are of particular interest for use in inverse scattering problems of realistic systems.

For both types of Darboux transformation we have determined associated integral equations via the symmetry property (9)–(12) [5]. The Darboux transformations correspond to the solutions of the integral equations with degenerate kernels. Because there are only two possibilities for constructing a symmetric degenerate kernel Q, its structure allows a simple classification of Darboux transformations. The original definition (4)–(6) comprises both classes.

Exactly solvable models which are directly related to the standard inverse scattering procedures of Marchenko, Gel'fand–Levitan and Burdet–Giffon, respectively, are obtained in a straightforward way from this set of Darboux transformations. Even more interesting is that type II transformations allow the derivation of equations suitable for the formulation of a matrix method. Thus, for inverse scattering problems at fixed energy the matrix method of Newton–Sabatier emerges as a Darboux transformation of type II with a specific choice of the functions $\zeta_0(r)$ and the integration range. Apart from these well established schemes, new exactly solvable models have also been obtained.

All inverse scattering formalisms for the half line which have been used in practical applications can be related to specific classes of the set of Darboux transformations discussed in this work. Our study of Darboux transformations also yields more general integral equations. However, they are of limited use as long as the symmetric kernels cannot be given in terms of the spectral information. In general, the incorporation of the spectral information in the symmetric kernel will be difficult and can only be accomplished in specific cases.

Summing up, we have studied Darboux transformations in a unified way, and we have constructed a wide class of exactly solvable models. Apart from further applications in standard inverse scattering problems the generalized scattering schemes will gain importance in the treatment of problems where k^2- and λ^2- dependent potentials are involved.

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